

A version of Fabry's theorem for power series with regularly varying coefficients

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Abstract

For real power series whose non-zero coefficients satisfy $|a_m|^{1/m} \rightarrow 1$, we prove a stronger version of Fabry's theorem relating the frequency of sign changes in the coefficients and analytic continuation of the sum of the power series.

For a set Λ of non-negative integers, we consider the counting function

$$n(x, \Lambda) = \#\Lambda \cap [0, x].$$

We say that Λ is *measurable* if the limit

$$\lim_{x \rightarrow +\infty} n(x, \Lambda)/x$$

exists, and call this limit the *density* of Λ .

Let $S = \{a_m\}$ be a sequence of real numbers. We say that a *sign change* occurs at the place m if there exists $k < m$ such that $a_m a_k < 0$ while $a_j = 0$ for $k < j < m$.

Theorem A. *The following two properties of a set Λ of positive integers are equivalent:*

(i) *Every power series*

$$f(z) = \sum_{m=0}^{\infty} a_m z^m \tag{1}$$

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of radius of convergence 1, with real coefficients and such that the changes of sign of $\{a_m\}$ occur only for $m \in \Lambda$, has a singularity on the arc

$$I_\Delta = \{e^{i\theta} : |\theta| \leq \Delta\},$$

and

(ii) For every $\Delta' > \Delta$ there exists a measurable set $\Lambda' \subset \mathbf{N}$ of density Δ' such that $\Lambda \subset \Lambda'$.

Implication (ii) \longrightarrow (i) is a consequence of Fabry's General Theorem [6, 3], as restated by Pólya. For the implication (i) \longrightarrow (ii) see [9]. Fabry's General theorem takes into account not only the sign changes of coefficients but also the absolute values of coefficients. It has a rather complicated statement and the sufficient condition of the existence of a singularity given by this theorem is not the best possible. The best possible condition in Fabry's General theorem is unknown, see, for example the discussion in [4].

Alan Sokal (private communication) asked what happens if we assume that the power series (1) satisfies the additional regularity condition:

$$\lim_{m \in P, m \rightarrow \infty} |a_m|^{1/m} = 1, \quad (2)$$

where $P = \{m : a_m \neq 0\}$. This condition holds for most interesting generating functions. The answer is somewhat surprising:

Theorem 1. *The following two properties of a set Λ of positive integers are equivalent:*

a) *Every power series (1) satisfying (2), with real coefficients and such that the changes of sign of the coefficients a_m occur only for $m \in \Lambda$, has a singularity on the arc I_Δ , and*

b) *All measurable subsets $\Lambda' \subset \Lambda$ have densities at most Δ .*

We recall that the *minimum density*

$$D_2(\Lambda) = \lim_{r \rightarrow 0+} \liminf_{x \rightarrow +\infty} \frac{n((r+1)x, \Lambda) - n(x, \Lambda)}{rx}$$

can be alternatively defined as the sup of the limits

$$\lim_{x \rightarrow \infty} n(x, \Lambda')/x \quad (3)$$

over all measurable sets $\Lambda' \subset \Lambda$.

Similarly the *maximum density* of Λ is

$$\overline{D}_2(\Lambda) = \lim_{r \rightarrow 0+} \limsup_{x \rightarrow \infty} \frac{n((r+1)x, \Lambda) - n(x, \Lambda)}{rx},$$

and it equals to inf of the limits (3) over all measurable sequences of non-negative integers Λ' containing Λ .

For all these properties of minimum and maximum densities see [12].

Thus condition (ii) is equivalent to $\overline{D}_2(\Lambda) \leq \Delta$ while condition b) is equivalent to $\underline{D}_2(\Lambda) \leq \Delta$.

Corollary 1. *The following two properties of a set Λ of positive integers are equivalent:*

A. *Every power series*

$$\sum_{m \in \Lambda} a_m z^m \tag{4}$$

satisfying (2) has a singularity on I_Δ ,

A'. *Every power series (4) satisfying (2) has a singularity on every closed arc of length $2\pi\Delta$ of the unit circle, and*

B. $\underline{D}_2(\Lambda) \leq \Delta$.

Indeed, all assumptions of A are invariant with respect to the change of the variable $z \mapsto \lambda z$, $|\lambda| = 1$, thus A is equivalent to the formally stronger statement A'.

Now, the number of sign changes of any sequence does not exceed the number of its non-zero terms, thus B implies A by Theorem 1. The remaining implication $A \longrightarrow B$ will be proved in the end of the proof of Theorem 1.

Proof of Theorem 1. b) \longrightarrow a). Proving this by contradiction, we assume that $\underline{D}_2(\Lambda) \leq \Delta$, and there exists a function f of the form (1) with the property (2) which has an analytic continuation to I_Δ , and such that the sign changes occur only for $m \in \Lambda$.

Without loss of generality we assume that $a_0 = 1$, and $\Delta < 1$.

Lemma 1. *For a function f as in (1) to have an immediate analytic continuation from the unit disc to the arc I_Δ it is necessary and sufficient that there exists an entire function F of exponential type with the properties*

$$a_m = (-1)^m F(m), \quad \text{for all } m \geq 0, \tag{5}$$

and

$$\limsup_{t \rightarrow \infty} \frac{\log |F(te^{i\theta})|}{t} \leq \pi b |\sin \theta|, \quad |\theta| < \alpha, \quad (6)$$

with some $b < 1 - \Delta$.

This result can be found in [1], see also [2, 4].

Consider the sequence of subharmonic functions

$$u_m(z) = \frac{1}{m} \log |F(mz)|, \quad m = 1, 2, 3, \dots$$

This sequence is uniformly bounded from above on every compact subset of the plane, because F is of exponential type. Moreover, $u_m(0) = 0$ because of our assumption that $a_0 = F(0) = 1$. Compactness Principle [8, Th. 4.1.9] implies that from every sequence of integers m one can choose a subsequence such that the limit $u = \lim u_m$ exists. This limit is a subharmonic function in the plane that satisfies in view of (6)

$$u(re^{i\theta}) \leq \pi br |\sin \theta|, \quad |\theta| < \alpha, \quad (7)$$

with some b satisfying

$$0 < b < 1 - \Delta.$$

We use the following result of Pólya [11, footnote 18, p. 703]:

Lemma 2. *Let f be a power series (1) of radius of convergence 1. Let $\{a_{m_k}\}$ be a subsequence of coefficients with the property*

$$\lim_{k \rightarrow \infty} |a_{m_k}|^{1/m_k} = 1,$$

and assume that for some $r > 0$ the number of non-zero coefficients a_j on the interval $m_k \leq j \leq (1+r)m_k$ is $o(m_k r)$ as $k \rightarrow \infty$. Then f has no analytic continuation to any point of the unit circle.

Lemma 2 also follows from the results of [1] or [4].

Now we show that (2) implies the following:

Lemma 3. *Every limit function has the property $u(x) = 0$ for $x \geq 0$.*

Proof of Lemma 3. Let $U = \{x : x \geq 0, u(x) < 0\}$. This set is open because u is upper semi-continuous. Take any closed interval $J = [c, d] \subset U$.

Then $u(x) \leq -\epsilon$, $x \in J$, with some $\epsilon > 0$. Let $\{m_k\}$ be the sequence of integers such that $u_{m_k} \rightarrow u$. Then from the definition of u_m we see that

$$\log |F(m_k x)| \leq -m_k \epsilon / 2 \quad \text{for } x \in J$$

and for all large k . Together with (5) and (2) this implies that $a_j = 0$ for all $j \in m_k J$. Let $a_{m'_k}$ be the last non-zero coefficient before cm_k . Applying Lemma 2 to the sequence $\{m'_k\}$ we conclude that f has no analytic continuation from the unit disc. This is a contradiction which proves Lemma 3. \square

Now we use the following general fact:

Grishin's Lemma. *Let $u \leq v$ be two subharmonic functions, and μ and ν their respective Riesz measures. Let E be a Borel set such that $u(z) = v(z) > -\infty$ for $z \in E$. Then the restrictions of the Riesz measures on E satisfy*

$$\mu|_E \leq \nu|_E.$$

The references are [13, 7, 5].

In view of Lemma 2, we can apply Grishin's Lemma to u and $v(z) = \pi b |\operatorname{Im} z|$ and $E = [0, \infty) \subset \mathbf{R}$. We obtain that the Riesz measure $d\mu$ of any limit function u of the sequence $\{u_k\}$ satisfies

$$d\mu|_{[0, \infty)} \leq b \, dx. \tag{8}$$

Now we go back to our coefficients and function F . By our assumption, the sign changes occur on a sequence Λ whose minimum density is at most Δ . Choose a number a such that $b < a < 1 - \Delta$. By the first definition of the minimum density, there exist $r > 0$ and a sequence $x_k \rightarrow \infty$ such that

$$n((1+r)x_k, \Lambda) - n(x_k, \Lambda) \leq (1-a)rx_k.$$

Lemma 4. *Let (a_0, a_1, \dots, a_N) be a sequence of real numbers, and f a real analytic function on the closed interval $[0, N]$, such that $f(n) = (-1)^n a_n$. Then the number of zeros of f on $[0, N]$, counting multiplicities, is at least N minus the number of sign changes of the sequence $\{a_n\}$.*

Proof. Consider first an interval (k, n) such that $a_k a_n \neq 0$ but $a_j = 0$ for $k < j < n$. We claim that f has at least

$$n - k - \#(\text{sign changes in the pair } (a_k, a_n))$$

zeros on the open interval (k, n) . Indeed, the number of zeros of f on this interval is at least $n - k - 1$ in any case. This proves the claim if there is a sign change in the pair (a_k, a_n) . If there is no sign change, that is $a_n a_k > 0$, then $f(n)f(k) = (-1)^{n-k}$. So the number of zeros of f on the interval (n, k) is of the same parity as $n - k$. But f has at least $n - k - 1$ zeros on this interval, thus the total number of zeros is at least $n - k$. This proves our claim.

Now let a_k be the first and a_n the last non-zero term of our sequence. As the interval (k, n) is a disjoint union of the intervals to which the above claim applies, we conclude that the number of zeros of f on (k, n) is at least $(n - k)$ minus the number of sign changes of our sequence. On the rest of the interval $[0, N]$ our function has at least $N - n + k$ zeros, so the total number of zeros is at least N minus the number of sign changes. \square

Let u be a limit function of the subsequence $\{u_{m_k}\}$ with $m_k = [x_k]$. By Lemma 4, the function F has at least $arx_k - 2$ zeros on each interval $[x_k, (1 + r)x_k]$, which implies that the Riesz measure μ of u satisfies

$$\mu([1, 1 + r]) \geq ar.$$

This contradicts (8) and thus proves the implication b) \longrightarrow a).

a) \longrightarrow b). Suppose that a set Λ of positive integers does not satisfy b). We will construct power series f of the form (4) which has an immediate analytic continuation from the unit disc to the arc I_Δ . This will simultaneously prove the implications a) \longrightarrow b) of Theorem 1 and $A \longrightarrow B$ of Corollary 1.

Let $\Lambda' \subset \Lambda$ be a measurable set of density $\Delta' > \Delta$. Let S be the complement of Λ' in the set of positive integers. Then S is also measurable and has density $1 - \Delta'$.

Consider the infinite product

$$F(z) = \prod_{t \in S} \left(1 - \frac{z^2}{t^2}\right).$$

This is an entire function of exponential type with indicator $\pi(1 - \Delta')|\sin \theta|$, and furthermore,

$$\log |F(z)| \geq \pi(1 - \Delta')|\operatorname{Im} z| + o(|z|), \quad (9)$$

as $z \rightarrow \infty$ outside the set $\{z : \operatorname{dist}(z, S) \leq 1/4\}$. (See [10, Ch. II, Thm. 5] for this result.) Now we use the sufficiency part of Lemma 1, and define

the coefficients of our power series by $a_m = (-1)^m F(m)$. Then we have all needed properties, in particular (2) follows from (9).

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References

- [1] N. U. Arakelyan and V. A. Martirosyan, Localization of singularities on the boundary of the circle of convergence, *Izvestiya Akademii Nauk Armyanskoi SSR, Mat.* vol. 22, No. 1 (1987) 3-21 (Russian) English translation: *Journal of Contemporary Mathematical Analysis*, 22 (1988) 1–19.
- [2] N. Arakelyan, W. Luh and J. Müller, On the localization of singularities of lacunar power series, *Complex Variables and Elliptic Equations*, 52 (2007) 651–573.
- [3] L. Bieberbach, *Analytische Fortsetzung*, Springer, Berlin, 1955.
- [4] A. Eremenko, Densities in Fabry's theorem, preprint arXiv:0709.2360.
- [5] B. Fuglede, Some properties of the Riesz charge associated with a δ -subharmonic function. *Potential Anal.* 1 (1992) 355–371.
- [6] E. Fabry, Sur les séries de Taylor qui ont une infinité de points singuliers, *Acta math.*, 22 (1898) 65–87.
- [7] A. F. Grishin, Sets of regular growth of entire functions I, *Teor. Funktsii Funktsional. Anal. i Prilozhen.* No. 40 (1983) 36–47 (Russian).
- [8] L. Hörmander, *The analysis of linear partial differential operators*, vol. I, Springer, Berlin, 1983.
- [9] P. Koosis, *The Logarithmic Integral*, vol. II Cambridge Univ. Press, Cambridge, 1992.
- [10] B. Ya. Levin, *Distribution of zeros of entire functions*, AMS Providence, RI, 1980.

- [11] G. Pólya, Über gewisse notwendige Determinantkriterien für die Fortsetzbarkeit einer Potenzreihe, Math. Ann. 99 (1928) 687–706.
- [12] G. Pólya, Untersuchungen über Lücken und Singularitäten von Potenzreihen, Math. Z., 29 (1929) 549–640.
- [13] C. de la Valle-Poussin, Potentiel et problème généralisé de Dirichlet, Math. Gazette, 22 (1938) 17–36.

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